Let us consider the two homogeneous linear equations

$$a_1x + b_1y = 0$$
$$a_2x + b_2y = 0$$

Multiplying the first equation by b_2, the second by b_1, substracting and dividing by $x\,(\neq 0)$, we get

$$a_1b_2 - a_2b_1 = 0$$

This is conveniently written as

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$$

The expresion on the left is called a Determinant, Which consists of two rows and two columns. So, this is called a second order Determinant; a_1, b_1, a_2 and b_2 are called element of the determinant.

Next, we consider the three homogenous linear equations

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$
$$a_3x + b_3y + c_3z = 0$$

Eliminating x, y, z from these three we get

$$a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - c_2a_3) + c_1(a_2b_3 - a_3b_2) = 0$$

or, $$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0$$

This is convenienty written as

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

The expression on the left is called a Determinant. As it consists of three rows and three columns, it is called a Third Order Determinant.

We first define the determinant of order 2 and 3 and next we complete this by defining 1st order determinant.

Determinant of order 2 (Second order Determinant)

For a 2nd order matrix $A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ its determinant is the number $\text{Det}(A) = |A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - b_1a_2$

Determinant of order 3 (Third order Determinant)

For a 3rd order matrix $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$

its determinant is ~~the number~~.

$$\text{Det}(A) = |A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - c_2b_3) - b_1(a_2c_3 - c_2a_3) + c_1(a_2b_3 - a_3b_2)$$

(By definition of 2nd order determinant)

Determinant of Order 1 (First order Determinant)

For a 1st Order matrix $A = (a_1)$ its determinant is the number $\text{Det}\,A = |A| = |a_1| = a_1$

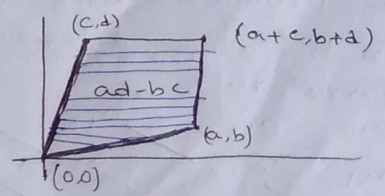
The determinant value is positive or negative

~~Min.~~ $\boxed{\text{Definition}}$:- In linear algebra, the determinant is a scalar value that can be computed from the elements of a square matrix and encodes certain properties of the linear transformation described by the matrix.

The determinant of a matrix A is denoted $\det(A)$ or $|A|$.

Geometrical Interpretation

2x2 matrices

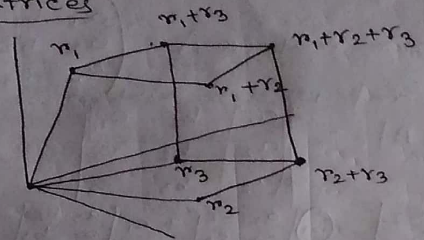


The area of the parallelogram is the absolute value of the determinant of the matrix formed by the vectors representing the parallelogram's sides.

The Leibniz formula for the determinant of a 2×2 matrix is $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

The absolute value of $ad-bc$ is the area of the parallelogram.

<u>3x3 matrices</u>



The volume of this parallelepiped is the absolute value of the determinant of the matrix formed by the rows constructed from the vectors r_1, r_2 and r_3.

(i) (3×3) matrix for the determinant value calculate Laplace formula.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

<u>Sarru's rule for Determinant</u>

The determinant for the 3 columns on the left is the sum of the products along the solid diagonals minus the sum of the product along the dashed diagonals.

In the above determinant value using the Sarru's rule

N.W → S.E

$$\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

Properties of Determinant

Following properties of determinants are true for the determinant of any order. However we give the proof for 3rd order determinant.

Property 1 The value of a determinant is unaltered if the determinant is transposed i.e. if rows and columns are interchanged; i.e.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Proof H.W.

Property 2 The value of a determinant is unaltered but the sign is altered if two rows or columns are interchanged i.e

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

(if 1st and 2nd row are considered to be interchanged; Symbolically we write interchanging R_1 and R_2)

Proof H.W.

Property 3 If two rows or columns of a determinant are identical then the value of the determinant is zero.

Proof Let D be the determinant whose ith row and jth row are identical. If those two rows are interchanged then the value of D is not changed but on the other hand, by the property 2 it becomes $-D$.

So, $D = -D$

$\therefore 2D = 0$ $\therefore D = 0$

Property 4. If all the elements of one row or column are multiplied by a number then the value of the determinant is multiplied by that number. i.e

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ xa_3 & xb_3 & xc_3 \end{vmatrix} = x \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Proof H.W.

Property 5 If each element of a row or a column is expressed as the sum of two numbers then the determinant can be expressed as sum of two determinants.

i.e.
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2+x & b_2+y & c_2+z \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ x & y & z \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(Considering the second row as sum of two numbers)

Proof
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2+x & b_2+y & c_2+z \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2+y & c_2+z \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2+x & c_2+z \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2+x & b_2+y \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3 + yc_3 - c_2b_3 - zb_3) - b_1(a_2c_3 + xc_3 - c_2a_3 - za_3) + c_1(a_2b_3 + xb_3 - b_2a_3 - ya_3)$$

$$= \{a_1(b_2c_3 - c_2b_3) - b_1(a_2c_3 - c_2a_3) + c_1(a_2b_3 - b_2a_3)\} + \{a_1(yc_3 - zb_3) - b_1(xc_3 - za_3) + c_1(xb_3 - ya_3)\}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ x & y & z \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Property 6 The value of determinant remains unaltered when to all the elements of any row (or column) the same multiple of the correspon ding elements of any other row (or column).

~~Proof~~

i.e.
$$\begin{vmatrix} a_1 + ma_2 + na_3 & b_1 + mb_2 + nb_3 & c_1 + mc_2 + nc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Proof ~~[illegible]~~ $= \begin{vmatrix} a_1+ma_2+na_3 & b_1+mb_2+nb_3 & c_1+mc_2+nc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} ma_2 & mb_2 & mc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} na_3 & nb_3 & nc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + m\begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + n\begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + m \cdot 0 + n \cdot 0 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Property 7

The sum of the products of elements of any row (or column) with the cofactor of the corresponding elements of some other row (or column) is zero.

i.e. $a_1A_2 + b_1B_2 + c_1C_2 = 0$

Proof let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$\therefore a_1A_2 + b_1B_2 + c_1C_2 = a_1 \times -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_1\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} + c_1 \times -\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

$$= -a_1(b_1c_3 - b_3c_1) + b_1(a_1c_3 - a_3c_1) - c_1(a_1b_3 - a_3b_1)$$

$$= -a_1b_1c_3 + a_1b_3c_1 + a_1b_1c_3 - a_3b_1c_1 - a_1c_1b_3 + a_3b_1c_1$$

$$= 0$$

Property 8 If each element in a row or in a column of a determinant is zero, then the value of the determinant is zero.

i.e. $\begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

Proof Expansion with 2nd row we get

$$\Delta = -0\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + 0\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - 0\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} = 0$$

Product of Two Determinants

let $\Delta_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}$

$\therefore \Delta = \Delta_1 \times \Delta_2$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \vec{\alpha_1} & \vec{\beta_1} \\ \alpha_2 & \beta_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix}$$

$(2\times2) \times (2\times2) = 2\times2$

let $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ $\quad \Delta_2 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$

$$\therefore \Delta = \Delta_1 \times \Delta_2 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

Illustration

Let $D = \begin{vmatrix} 3x^2 & 3x & 1 \\ x^2+2x & 2x+1 & 1 \\ 2x+1 & x+2 & 1 \end{vmatrix} = f(x)$

We see $f(1) = \begin{vmatrix} 3 & 3 & 1 \\ 3 & 3 & 1 \\ 3 & 3 & 1 \end{vmatrix}$ has three identical rows.

so $(x-1)^2$ must be a factor of the determinant D.

Minor of an element in a determinant

The sub determinant of a determinant det(A) obtained by deleting the rth row and sth column is called the Minor of the element belonging to the rth row and sth column of the determinant.

Example : (i) Let $D = \begin{vmatrix} 2 & 5 & 7 \\ 8 & 0 & 9 \\ 1 & 3 & 4 \end{vmatrix}$. Then the Minor of 8 (belonging to 2nd row and 1st column) $= \begin{vmatrix} 5 & 7 \\ 3 & 4 \end{vmatrix}$

(ii) Let $D = \begin{vmatrix} 6 & 4 \\ 1 & 3 \end{vmatrix}$. Then the Minor of 4 (the element belonging to first row and second column) $= |1| = 1$

Co-factor of an element in a determinant

If a is the element in a determinant lying in rth row and sth column then the co-factor of a in the determinant $= (-1)^{r+s} \times$ Minor of a in the determinant.

Example Let $D = \begin{vmatrix} 3 & 5 & 7 \\ 6 & 9 & 5 \\ 4 & 6 & 8 \end{vmatrix}$ Then co-factor of 5 (the element belonging to the 2nd row and 3rd column) is $(-1)^{2+3} \begin{vmatrix} 3 & 5 \\ 4 & 6 \end{vmatrix} = -(18-20) = 2$

Note: It is interesting to note that the minor and co-factor of an element does not depend on the value of the element rather it depends on its position in the determinant.

Theorem 1

If the elements of any row or column be multiplied by their corresponding co-factors then the sum of such products is equal to the determinant.

Proof

Let us consider a third order determinant

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Let the co-factors of a_1, b_1 and c_1 be A_1, B_1 and C_1 respectively.

We shall show i) $D = a_1A_1 + b_1B_1 + c_1C_1$

(ii) $D = a_1A_1 + a_2A_2 + a_3A_3$

(iii) $D = a_2A_2 + b_2B_2 + c_2C_2$

(iv) ~~and~~ $D = b_1B_1 + b_2B_2 + b_3B_3$

(v) $D = a_3A_3 + b_3B_3 + c_3C_3$

(vi) $D = c_1C_1 + c_2C_2 + c_3C_3$

(iv) **Proof**

$$B_1 = (-1)^{1+2} \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} = -(a_2c_3 - c_2a_3)$$

$$B_2 = (-1)^{2+2} \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} = (a_1c_3 - c_1a_3)$$

$$B_3 = (-1)^{3+2} \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = -(a_1c_2 - c_1a_2)$$

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - c_2b_3) - b_1(a_2c_3 - c_2a_3) + c_1(a_2b_3 - b_2a_3)$$

$$= b_1\{-(a_2c_3 - c_2a_3)\} + b_2(a_1c_3 - c_1a_3) + b_3\{-(a_1c_2 - c_1a_2)\}$$

$$= b_1B_1 + b_2B_2 + b_3B_3$$

Thus $D = b_1B_1 + b_2B_2 + b_3B_3$

Theorem 2

If the elements of any row or column be multiplied by the co-factors of the corresponding elements of a parallel row or column then the sum of such products is equal to 0.

Proof Let us consider a third order $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Let the co-factors of a_1, b_1 and c_1 are A_1, B_1 and C_1 respectively, we shall show

$$a_1A_2 + b_1B_2 + c_1C_2 = 0$$

$$a_2A_3 + b_2B_3 + c_2C_3 = 0$$

i.e. $a_iA_j + b_iB_j + c_iC_j = 0$ if $i \neq j$

and $a_1B_1 + a_2B_2 + a_3B_3 = 0$

$a_1C_1 + a_2C_2 + a_3C_3 = 0$ etc.

We are going to show any one of these say

$a_1B_1 + a_2B_2 + a_3B_3 = 0$

Now, $B_1 = (-1)^{1+2}\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} = -(a_2c_3 - c_2a_3)$

$B_2 = (-1)^{2+2}\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} = a_1c_3 - c_1a_3$

$B_3 = (-1)^{3+2}\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = -(a_1c_2 - c_1a_2)$

Then $a_1B_1 + a_2B_2 + a_3B_3$

$= a_1\{-(a_2c_3 - c_2a_3)\} + a_2\{a_1c_3 - \underline{c_1a_3}\} + a_3\{-(a_1c_2 - c_1a_2)\}$

$= -a_1a_2c_3 + a_1a_3c_2 + a_2a_1c_3 - a_3a_1c_2 - \underline{a_2c_1a_3} + a_3c_1a_2$

$= 0$

Example 1.

Find the value of $\begin{vmatrix} 17 & 58 & 97 \\ 19 & 60 & 99 \\ 18 & 59 & 98 \end{vmatrix}$

Solution $\begin{vmatrix} 17 & 58 & 97 \\ 19 & 60 & 99 \\ 18 & 59 & 98 \end{vmatrix} = \begin{vmatrix} 17 & 58 & 97 \\ 2 & 2 & 2 \\ 18 & 59 & 98 \end{vmatrix}$ [$R_2 - R_1$]

$= \begin{vmatrix} 17 & 58 & 97 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{vmatrix}$ [$R_3 - R_1$]

$= 2 \begin{vmatrix} 17 & 58 & 97 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$

$= 2 \times 0$ [$\because R_2$ and R_3 are identical] property 3

$= 0$

Example 2

Prove that $\begin{vmatrix} b+c & a-c & a-b \\ b-c & c+a & b-a \\ c-b & c-a & a+b \end{vmatrix} = 8abc$

Solution $\begin{vmatrix} b+c & a-c & a-b \\ b-c & c+a & b-a \\ c-b & c-a & a+b \end{vmatrix}$

$= \begin{vmatrix} b+c & a-c & a-b \\ 2b & 2a & 0 \\ c-b & c-a & a+b \end{vmatrix}$ [$R_2 - R_1$]

$= \begin{vmatrix} b+c & a-c & a-b \\ 2b & 2a & 0 \\ 2c & 0 & 2a \end{vmatrix}$ [$R_3 + R_1$]

$= 4 \begin{vmatrix} b+c & a-c & a-b \\ b & a & 0 \\ c & 0 & a \end{vmatrix}$ taking out 2 from R_2 and R_3

$= 4 \begin{vmatrix} b & a-c & a-b \\ b & a & 0 \\ c & 0 & a \end{vmatrix}$ [$R_1 - R_2$]

$= -4 \begin{vmatrix} 0 & c & b \\ b & a & 0 \\ c & 0 & a \end{vmatrix}$

$= -4\{0(a^2-0) - c(ab-0) + b(-ac)\} = 8abc$

Example 3

Prove without expanding, $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = 0$

Solution $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} \quad \text{[Property 5]}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \frac{abc}{abc}\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + \begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix} \quad \text{[interchanging } C_2 \text{ and } C_3\text{]}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \quad \text{[interchanging } C_1 \text{ and } C_2\text{]}$$

$= 0$

Example 4 Without expanding prove that

$$\underbrace{\begin{vmatrix} 0 & 6666 & 7777 \\ -6666 & 0 & 5555 \\ -7777 & -5555 & 0 \end{vmatrix}}_{A} = 0$$

Solution $D = |A|$

or, $D =$ [Taking (−1) out from each row]

or, $D =$ [Transposing]

or,

or $D = 0$

Example 5 Without expanding the determinant, prove that

$$\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix}$$

Solution (i) Interchanging C_2 and C_1

(ii) Interchanging R_1 and R_2

(iii) Transposing

Again (ii) → (i) → (iii)

Example 6

$$\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix}$$

$= (b-c)(c-a)(a-b)(a+b+c)(a^2+b^2+c^2)$

Solution (i) $C_2 - C_1$

(ii) $C_2 + 2C_3$

(iii) $C_2 + C_1$

(iv) common 2nd row

(v) $R_1 - R_2$

(vi) $R_2 - R_3$

(vii) ~~3rd column~~ [common $(a-b)$ from 1st row and $(b-c)$ from 2nd row]

(viii) $R_1 - R_2$

(ix) Common $(a-c)$ from 1st row

(x) Expansion

Example 7 Prove that

$$\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} = 2(b+c)(c+a)(a+b)$$

Solution ~~a+b+c~~

(i) $R_1 + R_2$

(ii) Common $(a+b)$ from 1st row

(iii) $R_2 + R_3$

(iv) Common ~~from~~ $(b+c)$ from 2nd row

(v) $C_3 + C_1$

(vi) Common $(c+a)$ from 3rd column

(vii) $R_1 + R_2$ (viii) Expansion

Example 8

$$\begin{vmatrix} (b+c)^2 & c^2 & b^2 \\ c^2 & (c+a)^2 & a^2 \\ b^2 & a^2 & (a+b)^2 \end{vmatrix} = 2(bc+ca+ab)^3$$

Solution L.H.S. (i) multiplication 1st row, 2nd row & 3rd row by a^2, b^2 & c^2 respectively.

(ii) Putting $x = bc$, $y = ca$, $z = ab$

(iii) same operation ~~previous~~ Example 9

Example 9 Prove that $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$

$= 2abc(a+b+c)$

Solution L.H.S. $= \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$

(i) $C_1 - C_3$ (ii) $C_2 - C_3$

(iii) Common $(a+b+c)$ from 1st row and 3rd row

(iv) $R_3 - R_1$

(v) $R_3 - R_2$

(vi) Common 2 from 3rd row and multiplication a & b 1st column & 3rd column

(vii) Common ab from 3rd row

(viii) $C_1 + C_3$ & $C_2 + C_3$

(ix) Expansion

Example 10 Show that $\begin{vmatrix} a^2 & 2ab & b^2 \\ b^2 & a^2 & 2ab \\ 2ab & b^2 & a^2 \end{vmatrix}$ is a perfect square.

Solution (i) $R_1 + R_2$ (ii) $R_1 + R_3$

(iii) Common $(a+b)^2$ from 1st row

(iv) $C_2 - C_1$ & then $C_3 - C_1$

(v) Expansion

Example 11

If ω be a cube root of unity, then prove that $(a+b\omega+c\omega^2)$ is a factor of the determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$. Find also another factors. Hence show that the determinant is equal to $-(a^3+b^3+c^3-3abc)$

Solution

2nd part

Since ω is a cube root of unity and cube root of unity has three values $1, \omega$ & ω^2 so $(a+b.1+c.1^2)$, $(a+b\omega+c\omega^2)$ and $(a+b\omega^2+c(\omega^2)^2)$

i.e. $(a+b+c)$, $(a+b\omega+c\omega^2)$ and $(a+b\omega^2+c\omega)$ are three factors of D. Now $(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$ is a three degree expression of a, b, c and D is also a three degree expression. So, we can write

$$D = K(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$$

Where K is a constant.

or, $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = K(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$

Equating the co-efficient of a^3 from both side we get $-1 = K$

$$\therefore \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$$

So, the other factors are $-(a+b+c)$ and $(a+b\omega^2+c\omega)$. Now we know $(a+b\omega+c\omega^2)(a+b\omega^2+c\omega) = a^2+b^2+c^2-ab-bc-ca$.

Hence the second part follows.

Example 12

If $D_r = \begin{vmatrix} r & x & \frac{n(n+1)}{2} \\ 2r-1 & y & n^2 \\ 3r-2 & z & \frac{n(3n-1)}{2} \end{vmatrix}$ then prove that

$D_1 + D_2 + D_3 + \cdots + D_n = 0$

Solution

Now $D_1 + D_2 + D_3 + \cdots + D_n$

$= \sum_{r=1}^{n} D_r = \sum_{r=1}^{n} \left| \quad \right|$

$= \begin{vmatrix} \sum_{r=1}^{n} r & x & \frac{n(n+1)}{2} \\ \sum_{r=1}^{n} (2r-1) & y & n^2 \\ \sum_{r=1}^{n} (3r-2) & z & \frac{n(3n-1)}{2} \end{vmatrix}$

1st row

$\frac{n(n+1)}{2}$

$2\sum r - \sum 1$

$= 2 \cdot \frac{n(n+1)}{2} - n$

$3\sum_{r=1}^{n} r - \sum_{r=1}^{n} 2$

$= 3 \cdot \frac{n(n+1)}{2} - 2n$

$= \begin{vmatrix} \frac{n(n+1)}{2} & x & \frac{n(n+1)}{2} \\ 2 \cdot \frac{n(n+1)}{2} - n & y & n^2 \\ 3 \cdot \frac{n(n+1)}{2} - 2n & z & \frac{n(3n-1)}{2} \end{vmatrix}$

$= \begin{vmatrix} \frac{n(n+1)}{2} & x & \frac{n(n+1)}{2} \\ n^2 & y & n^2 \\ \frac{n(3n-1)}{2} & z & \frac{n(3n-1)}{2} \end{vmatrix} = 0$ [Since C_1 & C_3 are identical]

Example 11

1st part

$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a+b\omega & b+c\omega & c+a\omega \\ b & c & a \\ c & a & b \end{vmatrix}$ doing $R_1 + \omega R_2$

$= \begin{vmatrix} a+b\omega+c\omega^2 & b+c\omega+a\omega^2 & c+a\omega+b\omega^2 \\ b & c & a \\ c & a & b \end{vmatrix}$ doing $R_1 + \omega^2 R_3$

$= \begin{vmatrix} a+b\omega+c\omega^2 & \omega^2(a+\frac{b}{\omega^2}+\frac{c}{\omega}) & \omega(a+b\omega+\frac{c}{\omega}) \\ b & c & a \\ c & a & b \end{vmatrix}$

$= \begin{vmatrix} a+b\omega+c\omega^2 & \omega^2(a+b\omega+c\omega^2) & \omega(a+b\omega+c\omega^2) \\ b & c & a \\ c & a & b \end{vmatrix}$

$= (a+b\omega+c\omega^2) \begin{vmatrix} 1 & \omega^2 & \omega \\ b & c & a \\ c & a & b \end{vmatrix}$

So, $(a+b\omega+c\omega^2)$ is a factor of the determinant D.

Product of two determinant

If two determinants of same order are multiplied then the product is a determinant of same order. There are two process of multiplication.

We show both for second and third order determinants.

For Second Order:

1st process (Row – Row wise multiplication)

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = \begin{vmatrix} a_1x_1 + b_1y_1 & a_1x_2 + b_1y_2 \\ a_2x_1 + b_2y_1 & a_2x_2 + b_2y_2 \end{vmatrix}$$

2nd process (Row – Column wise multiplication)

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = \begin{vmatrix} a_1x_1 + b_1x_2 & a_1y_1 + b_1y_2 \\ a_2x_1 + b_2x_2 & a_2y_1 + b_2y_2 \end{vmatrix}$$

For Third order

1st Process (Row – Row wise multiplication)

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1x_1 + b_1y_1 + c_1z_1 & a_1x_2 + b_1y_2 + c_1z_2 & a_1x_3 + b_1y_3 + c_1z_3 \\ a_2x_1 + b_2y_1 + c_2z_1 & a_2x_2 + b_2y_2 + c_2z_2 & a_2x_3 + b_2y_3 + c_2z_3 \\ a_3x_1 + b_3y_1 + c_3z_1 & a_3x_2 + b_3y_2 + c_3z_2 & a_3x_3 + b_3y_3 + c_3z_3 \end{vmatrix}$$

2nd process (Row – Column wise multiplication)

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1x_1 + b_1x_2 + c_1x_3 & a_1y_1 + b_1y_2 + c_1y_3 & a_1z_1 + b_1z_2 + c_1z_3 \\ a_2x_1 + b_2x_2 + c_2x_3 & a_2y_1 + b_2y_2 + c_2y_3 & a_2z_1 + b_2z_2 + c_2z_3 \\ a_3x_1 + b_3x_2 + c_3x_3 & a_3y_1 + b_3y_2 + c_3y_3 & a_3z_1 + b_3z_2 + c_3z_3 \end{vmatrix}$$

The above procedure can be proved by expanding both the sides. ~~we are leaving this as~~

Illustration (i) Row – Row wise multiplication is illustrated as:

$$\begin{vmatrix} 3 & 5 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} \begin{vmatrix} 6 & 2 & 3 \\ -2 & -3 & -4 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 3\times6+5\times2+1\times3 & 3\times(-2)+5\times(-3)+1\times(-4) & 3\times1+5\times1+1\times1 \\ 0\times6+1\times2+2\times3 & 0\times(-2)+1\times(-3)+2\times(-4) & 0\times1+1\times1+2\times1 \\ 1\times6+2\times2+3\times3 & 1\times(-2)+2\times(-3)+3\times(-4) & 1\times1+2\times1+3\times1 \end{vmatrix}$$

$$= \begin{vmatrix} 31 & -25 & 9 \\ 8 & -11 & 3 \\ 19 & -20 & 6 \end{vmatrix}$$

(ii) Row – Column wise multiplication is illustrated as:

$$\begin{vmatrix} 3 & 7 \\ 3 & -1 \end{vmatrix} \begin{vmatrix} 1 & -2 \\ 8 & 0 \end{vmatrix} = \begin{vmatrix} 3\times1+7\times8 & 3\times(-2)+7\times0 \\ 3\times1+(-1)\times8 & 3\times(-2)+(-1)\times0 \end{vmatrix} = \begin{vmatrix} 59 & -6 \\ -5 & -6 \end{vmatrix}$$

Adjoint or Adjugate Determinant

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ be a determinant, let,

A_i, B_i, C_i be the co-factors of a_i, b_i & c_i respectively (for $i = 1, 2, 3 \ldots$). Then the determinant

$$D' = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$ is called Adjoint or Adjugate determinant of D.

This definifition can be given for Second order determinant also.

Illustration If $D = \begin{vmatrix} a & 0 & c \\ a & b & 0 \\ 0 & b & c \end{vmatrix}$ then the co-factors of each of the elements are

⇒ The adjugate Determinant of D is

$$D' = \begin{vmatrix} bc & -ca & ab \\ bc & ca & -ab \\ -bc & ca & ab \end{vmatrix}$$

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Symmetric Determinant:

A determinant D is called symmetric if every element of ith row is identical as those of ith column for $i = 1, 2, 3, \ldots$. The elements of a symmetric determinant remain same if it is transposed.

For example, $D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ is a symmetric determinant of 3rd order.

$D = \begin{vmatrix} a & h \\ h & b \end{vmatrix}$ is a symmetric determinant of 2nd order.

Skew Symmetric Determinant:

A determinant D is called skew-symmetric if every element of ith row is of same magnitude but of opposite sign as those of ith column.

For example, $D = \begin{vmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{vmatrix}$ is a skew-symmetric determinant of order 3.

$D = \begin{vmatrix} 0 & h \\ -h & 0 \end{vmatrix}$ is a skew symmetric determinant of order 2.

Note: In a skew symmetric determinant the diagonal elements must be 0 because if x be a diagonal element then $x = -x$
or, $x = 0$

Theorem 1 The square of any determinant is a symmetric determinant.

Proof Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ be a determinant

Then $D^2 = \begin{vmatrix} a_1^2+b_1^2+c_1^2 & a_1a_2+b_1b_2+c_1c_2 & a_1a_3+b_1b_3+c_1c_3 \\ a_1a_2+b_1b_2+c_1c_2 & a_2^2+b_2^2+c_2^2 & a_2a_3+b_2b_3+c_2c_3 \\ a_1a_3+b_1b_3+c_1c_3 & a_2a_3+b_2b_3+c_2c_3 & a_3^2+b_3^2+c_3^2 \end{vmatrix}$

and this is a symmetric matrix.

Theorem 2 The adjugate of a symmetric determinant is symmetric.

Proof We consider third order determinant.

Let $D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ be a symmetric matrix

$\therefore$ Adjugate of $D = \begin{vmatrix} bc-f^2 & gf-ch & hf-bg \\ gf-ch & ca-g^2 & gh-af \\ hf-bg & gh-af & ab-h^2 \end{vmatrix}$

and we see this is symmetric.

Theorem 3 The adjugate of the third order skew symmetric determinant is symmetric.

$\Rightarrow$ H.W.

Theorem 4 The adjugate of the second order skew symmetric determinant is skew-symmetric.

$\Rightarrow$ Let $D = \begin{vmatrix} 0 & h \\ -h & 0 \end{vmatrix}$ be a second order skew symmetric determinant.

The adjugate determinant $D' = \begin{vmatrix} 0 & h \\ -h & 0 \end{vmatrix}$ which is also skew-symmetric.

Theorem 5 The value of a third order skew symmetric determinant is zero.

Proof Let $D = \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}$ be a 3rd order skew symmetric determinant.

Then $D = (-1)^3 \begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix}$ taking out (-1) from each column.

or, $D = \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}$ transposing

or, $D = -D$ or, $2D = 0$ or $D = 0$

Theorem 6 A second order skew symmetric determinant is a perfect square.

Proof Let $D = \begin{vmatrix} 0 & a \\ -a & 0 \end{vmatrix}$ be a second order skew symmetric determinant.

$\therefore D = 0 \times 0 - a \times (-a) = a^2$ which is a perfect square.

Note In general a skew symmetric determina of order n is 0 if n is odd and is perfect square if n is even.